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ArcGIS® Parcel Fabric: Least-Squares Adjustment

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Introduction

The ArcGIS® Parcel Fabric adjustment process is uniquely designed to test the integrity of a cadastral boundary network and to derive the most likely position for every point in the network. Part of that process uses a least-squares procedure with parcel dimensions treated as observations in a similar way to the adjustment of geodetic networks.

This paper explains the least-squares adjustment process in mathematical detail. Equations used in computing the adjustment are given without derivation; however, references to the standard literature are provided. The primary purpose is to describe the background, theory, and application of the least-squares method used within the ArcGIS parcel fabric technology.

The underlying mathematics of the least squares adjustment method is not widely known to the average land records expert, even those who make extensive use of parcel fabric technology. The least-squares adjustment is a powerful tool for assessing and improving the accuracy of parcel data held in GIS databases. This technical paper provides more specifics about least-squares, how the parcel fabric technology uses it, and thereby enhances the understanding of its application within the ArcGIS software.

Unique Characteristics of the Parcel Fabric Approach

Cadastral networks can be large and very complex. An initial analysis attempts to detect and report on data problems that could distort the adjustment. Typical errors checked for prior to adjustment are listed below.

- **General data errors**
  - No height data—the software assumes an elevation of zero.
  - From and To points of a line are the same.
  - There is a break in observational sequence of a line.
  - The coordinates of the From and To points are the same.
- **Bearing and distance measurement blunder check:** Checks that the difference between the bearing and distance computed from the initial coordinates and the measured values is less than a specified tolerance.
- **Close points check:** The close points check flags any points closer than a specified distance that do not have a line between them.

Cadastral networks possess unique characteristics, and some of these are exploited in formulating observation equations. For example, a cadastral survey may have a particular basis of bearing or azimuth reference that differs from that of the lines bounding each parcel, and consequently, each parcel can be treated like a direction set with an unknown orientation.

Distances stated on cadastral plans, while are not necessarily directly measured, do nonetheless reflect the measurements made by the surveyor in the field, and are derivations of those original field observations. Each distance on a cadastral plan can be

considered a ground distance, being the horizontal projection of the measured slope distance at the mean height of the line. Because the adjustment is carried out on a projection plane, ground distances must be converted to grid distances during the formulation of the observation equations. Since height is a parameter in this conversion, the height adopted for each line is defined based on the nearest control points, as described in more detail in section 2.3.1.

Parcel joining (see Parcel Editor documentation) generates initial coordinates for every point in the network, and these values may or may not be on the datum and projection of the control points. This process transforms the parcel point coordinates to the control system so that the magnitude of the corrections during the least-squares procedure is minimized. Observation equations are formed for each parcel bearing, distance, and line point. (For further description of line-points, see section 3.4.) Weights can be assigned automatically to all bearings and distances according to the date of survey, or these values can be assigned by the user. The normal equations are solved using a Cholesky decomposition procedure.

After completion of the least-squares adjustment, a few post-processing procedures can, optionally, be applied to enforce geometric constraints. These include enforcing line points and straight lines. When a line point is within the tolerance specified, it will be shifted back onto its corresponding boundary line. If it is outside the tolerance, a warning is written to the adjustment report. Enforcing straight lines retains the original subdivision structure. Often, a series of adjacent lots in a plan requires that front and/or back lot lines have the same bearing, meaning that the individual lot lines are intended to be collinear. Enforcing straight lines detects these plan structures and, if the boundary points are within the specified tolerance, will make these lines collinear. Unlike geodetic networks, cadastral networks may contain measurements that do not contribute to the adjustment results. These include isolated radiations or traverse lines connected to the network at only a single point. Such measurements are isolated from the adjustment process, and then used to re-compute points as a post-process.

1. Coordinate Transformation

1.1 Background

This section describes the technical procedure for transforming joined parcel fabric coordinates into the control coordinate system.

Parcel joining is the process of connecting common points from an unjoined parcel to their equivalents in the parcel fabric. Parcel joining generates the topology and connectivity of the fabric. Joining also uses a coordinate transformation between the unjoined parcel's local coordinate system and the projected coordinate system of the fabric. But because this transformation uses only the fabric coordinates and not control coordinates, inconsistencies can arise between fabric measurements and cadastral corner coordinates. To improve the consistency between control and fabric measurements, another transformation is applied just prior to performing a least-squares adjustment.

Coordinates of designated control points, along with their coordinate values obtained after the parcel joining process, are used together to determine transformation parameters between the control coordinate system and that of the joined parcel fabric. Transformation parameters are estimated using a two-dimensional linear conformal transformation or Helmert transformation. Conformal transformations preserve shape among the relative positions of the transformed points. This property makes them suitable for transforming surveys with point coordinates in two distinct coordinate systems into

one or the other of those systems.[1]* This is precisely what needs to be done to put a newly joined fabric into the coordinate system defined by the control points. Residuals after transformation are reported at each control point, and if these are within some pre-specified tolerance, the parcel fabric is transformed into the control coordinate system using the estimated parameters.[2] This process serves three purposes:

- Allows verification of the quality of the fabric points to be used in the least-squares adjustment
- Permits the transformation of the joined fabric into the control coordinate system
- Provides the initial approximate coordinates on parcel points that are needed to start the least-squares adjustment

1.2 Basic Theory

The Helmert transformation, also referred to as two-dimensional linear conformal transformation or four-parameter similarity transformation, does not allow for deformation of the space transformed. Shape is retained after transformation except for a possible uniform scaling.[3] Helmert transformations are characterized by[4]

- Scale—to create equal dimensions in the two coordinate systems
- Rotation—to make the reference axes of the two systems parallel
- Translation—to create a common origin for the two coordinate systems

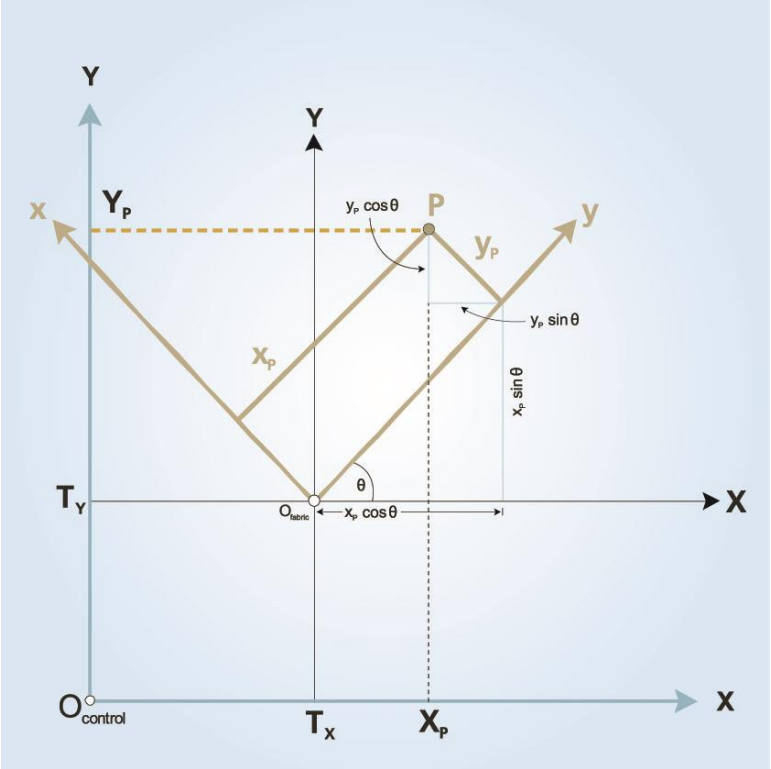
Scale and rotation are each defined by one parameter, and translation involves one parameter for each coordinate direction—for a total of four parameters. Having a minimum of two points with coordinates in both coordinate systems—called common points—permits a unique solution of the four parameters of the transformation. Having more than two common points allows estimation of the parameters by least squares. Once the values of the transformation parameters have been determined, points in the fabric coordinate system may be transformed into the control coordinate system.

1.3 Implementation

Figure 1 illustrates a two-dimensional Helmert transformation.

* Where bracketed numbers appear (e.g.,[1]), refer to complete citation in the Reference section of this document.

Figure 1
Geometry of the Two-Dimensional Helmert Transformation



The relationship between the control and parcel fabric coordinate systems is given by[3]

$$\begin{aligned} X_P &= S\left(x_p \cos\theta\right) - S\left(y_p \sin\theta\right) + T_X \\ Y_P &= S\left(x_p \sin\theta\right) + S\left(y_p \cos\theta\right) + T_Y \end{aligned} \quad , \tag{1.01}$$

where

X_p = x-coordinate (easting) in the control coordinate system
Y_p = y-coordinate (northing) in the control coordinate system
x_p = x-coordinate (easting) in the joined fabric coordinate system
y_p = y-coordinate (northing) in the joined fabric coordinate system
S = scale factor between the control and joined fabric coordinate systems
θ = rotation angle between the control and joined fabric coordinate systems.

Letting

$$a = S\cos\theta$$
$$b = S\sin\theta$$
$$c = T_X$$
$$d = T_Y$$

then (1.01) becomes

$$\text{(1.02)} \quad \begin{aligned} X_p &= ax_p - by_p + c \\ Y_p &= bx_p + ay_p + d \end{aligned},$$

where

$c = T_X$ translation in x-direction between the control and joined fabric coordinate systems
$d = T_Y$ translation in y-direction between the control and joined fabric coordinate systems.

The unknown coefficients to be estimated are *a, b, c,* and *d*. These four unknowns are functions of the transformation parameters *S,* θ, T_X, and T_Y, which can be deduced if desired. Since two equations can be written for every common point, two common points provide four equations and are sufficient for a unique solution. When more than two common points exist, a least-squares solution can be found. The observation equation form of (1.02) is[4]

$$\text{(1.03)} \quad \begin{aligned} ax_p - by_p + c &= X_p + v_{X_p} \\ bx_p + ay_p + d &= Y_p + +v_{Y_p}. \end{aligned}$$

For *n* common points (*n > 2; i = 1, 2,..., n*) there are *2n* coordinates from which a least-squares solution can be obtained according to the following algorithm:[5]

$$\text{(1.04a)} \quad \overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}, \quad \overline{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}, \quad \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \quad \overline{y} = \frac{\sum_{i=1}^{n} y_i}{n},$$

$$\text{(1.04b)} \quad \breve{X}_i = X_i - \overline{X}, \quad \breve{Y}_i = Y_i - \overline{Y}, \quad \breve{x}_i = x_i - \overline{x}, \quad \breve{y}_i = y_i - \overline{y},$$

$$\text{(1.04c)} \quad \hat{a} = \frac{\sum_{i=1}^{n} \breve{X}_i \breve{x}_i + \sum_{i=1}^{n} \breve{Y}_i \breve{y}_i}{\sum_{i=1}^{n} (\breve{x}_i^2 + \breve{y}_i^2)}, \quad \hat{b} = \frac{\sum_{i=1}^{n} \breve{Y}_i \breve{x}_i - \sum_{i=1}^{n} \breve{X}_i \breve{y}_i}{\sum_{i=1}^{n} (\breve{x}_i^2 + \breve{y}_i^2)},$$

(1.04d) $\hat{c} = \overline{X} - \hat{a}\overline{x} + \hat{b}\overline{y}, \quad \hat{d} = \overline{Y} - \hat{b}\overline{x} - \hat{a}\overline{y},$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ are the least-squares estimates of the transformation parameters a, b, c, and d. Adjusted coordinates are computed by substituting $\hat{a}$, $-\hat{b}$, $-\hat{c}$, and $\hat{d}$ into (1.02), that is,

(1.04e)
$$\begin{aligned} \hat{X}_i &= \hat{a}x_i - \hat{b}y_i + \hat{c} \\ \hat{Y}_i &= \hat{b}x_i + \hat{a}y_i + \hat{d}. \end{aligned}$$

Residuals are obtained from

(1.04f) $v_{X_i} = \hat{X}_i - X_i, \quad v_{Y_i} = \hat{Y}_i - Y_i.$

After adjustment, each control point used in estimating the parameters is quantitatively assessed and, depending on the outcome, accepted or rejected from the set of control points used for estimating the parameters. The testing process is as follows:
(1) transformation parameters are first computed using all the control points; (2) the residual at each common point is the difference between the computed values and the control values; (3) the point with the largest residual is excluded and the parameters are recomputed; (4) the standard error of the residuals of this new adjustment is computed; and (5) if the residuals of the excluded point are greater than three times this standard error, then the excluded point is rejected and the testing procedure applied again to the remaining control points.

This process continues until an excluded point passes the residual test. It is then included back into the adjustment and final transformation parameters are computed. The schematic algorithm below shows the iterative process of assessing the quality of control points.[6]

Control Point Quality Assessment - Schematic Algorithm

Step 1: Find the maximum residual after adjustment using $v_{MAX_k} = \max v_i \in \langle v_{X_i}, v_{Y_i} \rangle$, where *k* is the adjustment run index. Exclude the control point with the largest residual and recompute the transformation parameters using (1.04).

Step 2: Calculate the root-mean-square (RMS) of the residuals,

$$s_v = \left[\sum_{i=1}^{n^*} \left(v_{X_i}^2 + v_{Y_i}^2 \right) \Big/ 2n^* \right]^{1/2}.$$

Step 3: If $v_{MAX_k} > 3s_v$, then reject the offending control point and repeat steps 1 to 3.

Step 4: If $v_{MAX_k} < 3s_v$, then insert the excluded control point back into the set and recompute final transformation parameters.

Having obtained the least-squares estimates for the transformation parameters, the control system coordinates for the parcel fabric points can be computed by applying (1.04e), where X_i and Y_i are the control system coordinates for parcel points x_i and y_i.

2. Least-Squares Adjustment Mathematical Model

2.1 Preprocessing and Data Preparation

Section 1 described the process of transforming the parcel fabric coordinates into the control coordinate system. Upon successful completion of this step, a number of data preparation processes are performed prior to carrying out the least-squares adjustment on the parcel fabric measurements. These preparatory steps, briefly outlined in the introduction, are more fully described in Elfick (2005).[2]

2.2 Motivation and Basic Theory

Parcel fabric technology can be used to run a weighted least-squares adjustment to derive the most probable values of the coordinates of each parcel fabric point. The adjustment consists of two components: the functional model and the stochastic model. The functional model is a set of observation equations relating the measurements to the unknowns.

This functional model is based on a conformal map projection, the unknowns to be estimated being grid northing and easting. The stochastic model describes the expected errors associated with the measurements and, based on these, assigns each measurement a weight for use in the adjustment.

Least-squares theory, derivation of observation equations, matrix solution, and practical application are readily found in the literature as well as texts on surveying.[7,8,3,9,10,4] The reader is directed to these sources for theoretical background.

2.3 Functional Model

The parcel fabric technology uses a two-dimensional functional model based on a plane rectangular coordinate system. Heights of points are not considered within this model, but heights are needed during the preprocessing of observations. The table below lists the measurements adjusted and their general functional relationship to the unknowns.

Measurement	Functional Model	Unknowns
Distance	$d_{ij} = f(X_i, Y_i, X_j, Y_j)$	X_i = easting of point i
Direction	$b_{ij} = f(X_i, Y_i, X_j, Y_j, O_{ij})$	Y_i = northing of point i
True Mid-bearing	$a_{ij} = f(X_i, Y_i, X_j, Y_j)$	X_i = easting of point j
		Y_i = northing of point j
		O_{ij} = parcel orientation unknown

2.3.1 Distance Reductions

Distances are assumed to be horizontal and to be measured at the mean elevation of the area represented by the original record document. Since the least-squares adjustment takes place on the projection plane (grid surface) defined by the spatial reference system of the dataset, all distances must undergo a reduction to the projection plane. Reduction of measured horizontal distances to the grid projection plane is a two-step process. First, the horizontal distance is reduced to the reference ellipsoid using the computed elevation factor (also called sea level reduction), then this geodetic distance is further reduced to a grid distance using the point scale factors computed at the endpoints of the line. Grid distance is obtained from[11]

(2.01) $d_G = d_H k_e k_l = k d_H$,

where

$d_G =$ grid distance
$d_H =$ measured horizontal distance
$k_e =$ elevation factor
$k_l =$ line scale factor
$k = k_e k_l =$ combined scale factor.

The elevation factor k_e is computed from

(2.02) $k_e = \frac{R}{R+h} = \frac{R}{R+H+N}$,

where

$R =$ mean radius of the earth
$h =$ height above the reference ellipsoid
$H =$ orthometric height
$N =$ geoid undulation.

In (2.01), the line scale factor k_l is derived from the mean point scale factors at the terminal ends of the line, k_{p_1}, and k_{p_2}, that is,

(2.03) $k_l = \frac{k_{p_1} + k_{p_2}}{2}$.

Point scale factors depend on the reference ellipsoid and map projection used to define the spatial reference system. The Esri Projection Engine computational routines allow any supported map projection to be used as the grid surface for the parcel fabric adjustment. Given projection (grid) coordinates for the fabric points, the following schematic algorithm is implemented for computing elevation factor k_e and line scale factor k_l.

Scale Factor - Schematic Algorithm

Step 1: Transform the *(X, Y)* coordinates of the line endpoints into latitude and longitude using Esri Projection Engine routines, *(PCS)⇒(GCS)*.

Step 2: For each endpoint, compute mean radius of curvature R_φ at latitude φ (e.g., equation (3.45) in Lauf [1983]).[13]

Scale Factor - Schematic Algorithm (continued)

Step 3: Compute elevation factor k_e using (2.02) and R_φ from step 2. In (2.02), the value for *h* is determined based on the control points nearest to the line in question, as described in the next schematic algorithm.

Step 4: Compute ellipsoidal distance s_{12} using Robbins' geodetic direct problem algorithm (e.g., page 64 in Lauf [1983]).[13]

Step 5: Compute grid distance using $d_G^o = \sqrt{\left(N_2^o - N_1^o\right)^2 + \left(E_2^o - E_1^o\right)^2}$, where N_i^o, E_i^o are approximate grid coordinates from the coordinate transformation described in section 1.3, taking $X_i = E_i^o$ and $Y_i = N_i^o$.

Step 6: Compute line scale factor using $k_l = \frac{d_G^o}{s_{12}}$.'

Approximate grid coordinates for the terminal endpoints come from the coordinate transformation applied to the points (see section 1.3) and are satisfactory as arguments for computing the line scale factor.

The elevation factor given by (2.02) depends on the spatial reference system since the mean radius of the earth, ellipsoidal height, and geoid undulation all depend on the reference ellipsoid chosen. The fabric adjustment determines the approximate height for input in (2.02) from all the heights of the control points used in the adjustment. The height is computed for each line by interpolation using inverse distance weighting from the control points to the mid-point of the measured line in question.

The following schematic algorithm presents the methodology used for interpolating the elevation of a line from the control point elevations.

Height Interpolation - Schematic Algorithm

Step 1: Compute the position of point *P* midway along the line.

Step 2: For each control point *C* with elevation $Z > 0$, compute distance *d* between *P* and *C*; the interpolated height at *P*, h_p is found as follows:

$$T = \sum_{i=1}^{n} \frac{Z_i}{d_i^{\,2}}, \quad W = \sum_{i=1}^{n} \frac{1}{d_i^{\,2}}, \quad \Rightarrow \quad h_P = T/W$$

The correct height to use on the control points in computing the elevation factor is an ellipsoidal height, not an orthometric height. Since ellipsoidal heights can differ from orthometric heights by up to 100 meters, using the orthometric height of a control point in (2.02) will yield an incorrect elevation factor.

When distances are reduced using (2.02) with orthometric heights instead of ellipsoidal heights, the error in reduced distance is given by[12]

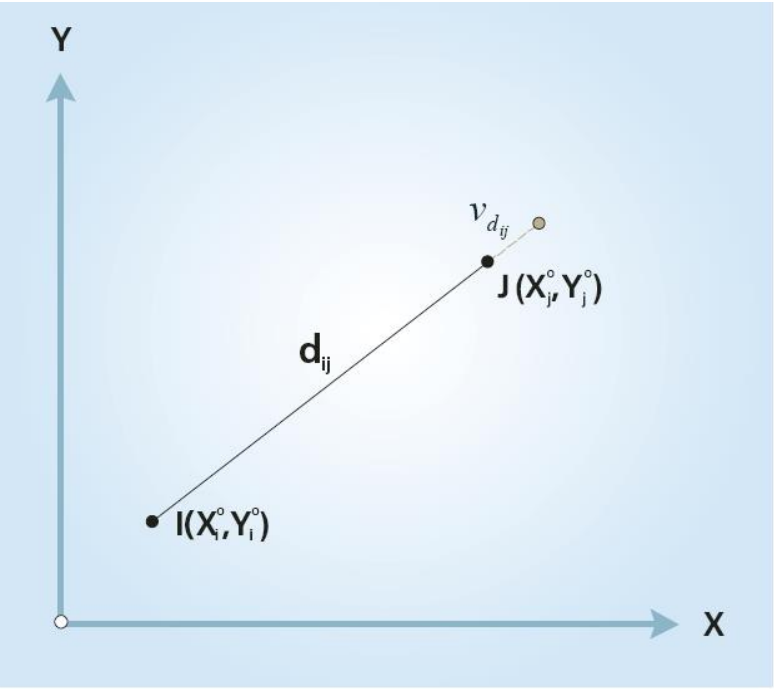
(2.04) $\Delta d_G = \frac{N_1 + N_2}{2R} d_G.$

This equation quantifies the effect of neglecting geoid undulation in reducing distances to the projection plane. The error Δd_G affects the scale of the parcel fabric being adjusted. For example, taking $N_1 = N_2 = 30$ m and $R = 6,371$ km, (2.04) gives a scale error of about 5 ppm (5 mm per km).

2.3.2 Distance Observation Equation

Distance observation equations relate measured lengths and their inherent random errors to the most probable coordinates for their endpoints[7] (refer to figure 2).

Figure 2
Distance Observation Equation



The following observation equation may be written for any line *IJ*:[1,9]

(2.05) $d_{ij} + v_{d_{ij}} = \sqrt{(\hat{X}_j - \hat{X}_i)^2 + (\hat{Y}_j - \hat{Y}_i)^2}$,

where d_{ij} is the measured grid distance of the line between points *I* and *J*, $v_{d_{ij}}$ the measurement residual, $(\hat{X}_i, \hat{Y}_i)$ the adjusted grid coordinate values for parcel point *I*, and $(\hat{X}_j, \hat{Y}_j)$ the adjusted grid coordinate values for parcel point *J*. The linearized distance observation equation is given by

$$(2.06)\quad v_{d_{ij}} + \left(\frac{X_i^o - X_j^o}{d_{ij}^o}\right)\Delta X_i + \left(\frac{Y_i^o - Y_j^o}{d_{ij}^o}\right)\Delta Y_i + \left(\frac{X_j^o - X_i^o}{d_{ij}^o}\right)\Delta X_j + \left(\frac{Y_j^o - Y_i^o}{d_{ij}^o}\right)\Delta Y_j = f_{d_{ij}}\cdot$$

In (2.06), $X_i^o, Y_i^o, X_j^o, \text{and } Y_j^o$ are initial approximations of the unknowns $X_i, Y_i, X_j, \text{and } Y_j$, computed after application of the final transformation parameters in (1.04) (see section 1.3);

$$d_{ij}^o = \sqrt{(X_j^o - X_i^o)^2 + (Y_j^o - Y_i^o)^2}\ ;$$

$$(2.06a)\quad f_{d_{ij}} = d_{ij}^o - d_{ij}\ ;$$

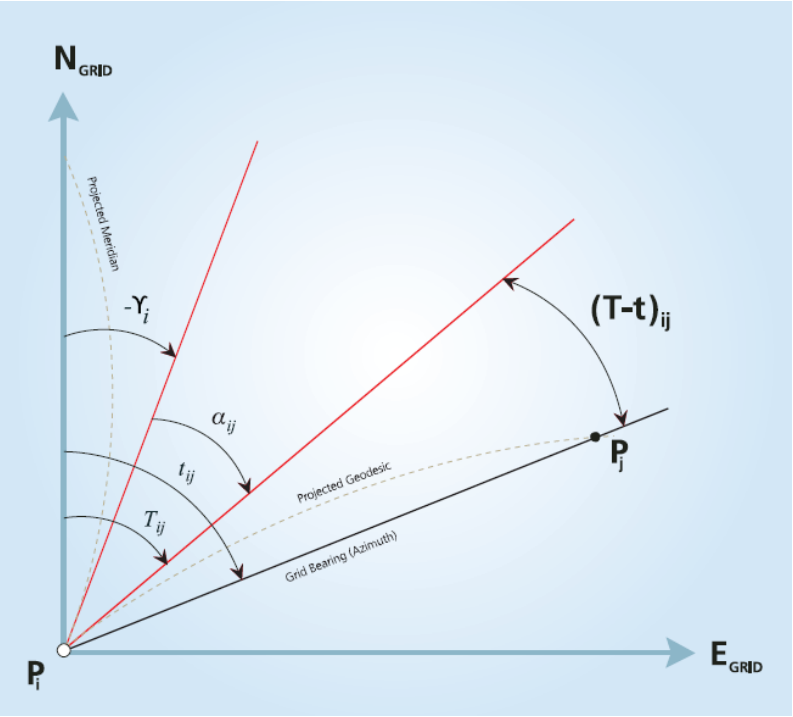
and $\Delta X_i, \Delta Y_i, \Delta X_j, \text{and } \Delta Y_j$ are corrections to be applied to the initial approximations such that

$$(2.06b)\quad \begin{aligned} \hat{X}_i &= X_i^o + \Delta X_i \\ \hat{Y}_i &= Y_i^o + \Delta Y_i \\ \hat{X}_j &= X_j^o + \Delta X_j \\ \hat{Y}_j &= Y_j^o + \Delta Y_j . \end{aligned}$$

2.3.3 Bearing Reductions

A geodesic on the surface of the ellipsoid projects as a curved line on a conformal map projection, with its concave side facing the central meridian.[13] The difference between the azimuth of the tangent to the projected geodesic of a line and the grid azimuth of the same line is called the curvature correction,[14] also known as arc-to-chord, T-t, second-term, or second-difference correction. The magnitude of T-t depends on the projection, direction, and length of the line and its location within a grid zone. Maximum T-t occurs for long lines located near the edge of zones and oriented parallel with the projection's standard lines.[11] Calculation of T-t also depends on the projection used. The parcel fabric technology calculates T-t for Universal Transverse Mercator (UTM) and Lambert Conformal Conic (LCC) projections. Since bearings are assumed to be grid bearings, the T-t correction is applied to all measured angles when converting them into grid bearings, and both T-t and meridian convergence corrections are applied to measured azimuths when converting them into grid bearings. These concepts are illustrated in figure 3 below.

Figure 3
Meridian Convergence and (T-t) Correction



The bearing of a line differs at each point along the line, so for very long lines, true mid-bearings are often used in cadastral surveys. A true mid-bearing is the grid azimuth of a line plus the grid convergence at its midpoint. To obtain a true mid-bearing on a curved surface, half the meridian convergence of the line needs to be applied at one end. The fabric technology uses the Esri Projection Engine routines to support any map projection used as the grid surface for the parcel fabric. Given projection (grid) coordinates for the fabric points, the following schematic algorithm is implemented for handling true mid-bearings:

Conversion of True Mid-bearing to Grid Bearing - Schematic Algorithm

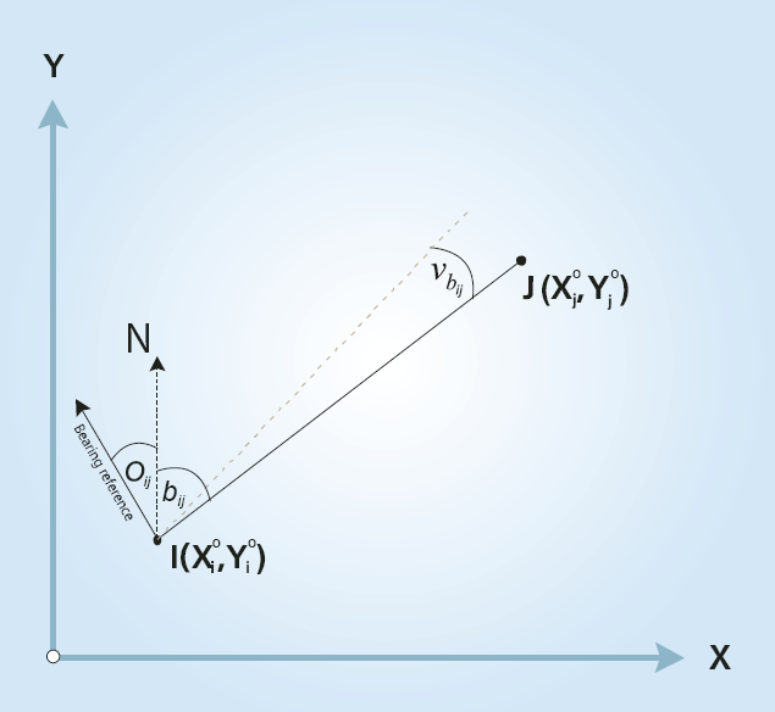
Step 1: Compute meridian convergence γ_1 and γ_2 at each endpoint using ESRI Projection Engine routines, *(PCS)⇒(GCS)*.

Step 2: Compute grid azimuth t_G using $t_G = t_m - \left(\frac{\gamma_1 + \gamma_2}{2} \right)$, where t_m is the measured true mid-bearing between the endpoints of the line.

2.3.4 Bearing Observation Equation

Bearings are usually not observed directly but derived either from measured angles, GPS baselines, or some other combination of field survey measurements. Bearing observation equations relate measured (or derived) bearings and their inherent random errors to the most probable coordinates for their endpoints [7] (refer to figure 4).

Figure 4
Bearing Observation Equation



The following observation equation may be written for any bearing line *IJ*:[4]

$$(2.07)\quad b_{ij} + v_{b_{ij}} = \tan^{-1}\left(\frac{\hat{X}_j - \hat{X}_i}{\hat{Y}_j - \hat{Y}_i}\right),$$

where b_{ij} is the measured bearing of the line between points *I* and *J*; $v_{b_{ij}}$ is the measurement residual; $(\hat{X}_i, \hat{Y}_i)$ is the adjusted grid coordinate values for parcel point *I;* and $(\hat{X}_j, \hat{Y}_j)$ is the adjusted grid coordinate values for parcel point *J*. The correct quadrant for the bearing b_{ij} is determined by the sign of the numerator and denominator in the tangent function argument.

The fabric adjustment treats the bearings of each parcel similar to a direction set where an unknown orientation is included in the bearing observation equations for each parcel. In this context, a direction set is defined as a collection of observations defined by the bearings of the parcel lines. The parcel bearings are correct with respect to each other, but the collection of parcel lines as a whole may be rotated with respect to the grid. This rotation is the orientation unknown to be calculated for the parcel. The orientation unknown is the correction that should be applied to each bearing in the collection to align it with its grid bearing as closely as possible. The approximate orientation is computed from the transformation parameters estimated during parcel joining (see section 1.1). That is,

(2.08) $$O_{ij}^{o} = \tan^{-1}\left(\frac{\hat{b}}{\hat{a}}\right),$$

where $\hat{a}$ and $\hat{b}$ are the estimated transformation parameters obtained when the parcel is joined into the existing fabric. The fabric least-squares adjustment estimates a correction to be applied to the initial orientation approximation such that

(2.09) $$\hat{O}_{ij} = O_{ij}^{o} - \Delta O_{ij}.$$

The linearized bearing observation equation is given by

(2.10) $$v_{b_{ij}} + \left(\frac{Y_i^o - Y_j^o}{\left(d_{ij}^o\right)^2}\right)\Delta X_i + \left(\frac{X_j^o - X_i^o}{\left(d_{ij}^o\right)^2}\right)\Delta Y_i + \left(\frac{Y_j^o - Y_i^o}{\left(d_{ij}^o\right)^2}\right)\Delta X_j + \left(\frac{X_i^o - X_j^o}{\left(d_{ij}^o\right)^2}\right)\Delta Y_j - \Delta O_{ij} = f_{b_{ij}},$$

where $X_i^o, Y_i^o, X_j^o, Y_j^o, X_i, Y_i, X_j, Y_j, \Delta X_i, \Delta Y_i, \Delta X_j,$ and ΔY_j are defined as in equation (2.06)

$$\left(d_{ij}^o\right)^2 = (X_j^o - X_i^o)^2 + (Y_j^o - Y_i^o)^2,$$

(2.10a) $$f_{b_{ij}} = b_{ij}^o - b_{ij},$$ and

$$b_{ij}^o = \tan^{-1}\left(\frac{X_j^o - X_i^o}{Y_j^o - Y_i^o}\right).$$

2.3.5 True Midbearing and Azimuth Observation Equations

Since a true mid-bearing is really an azimuth, it does not contain an orientation term. Therefore, the linearized true mid-bearing or azimuth observation equation is given by

(2.11) $$v_{a_{ij}} + \left(\frac{Y_i^o - Y_j^o}{\left(d_{ij}^o\right)^2}\right)\Delta X_i + \left(\frac{X_j^o - X_i^o}{\left(d_{ij}^o\right)^2}\right)\Delta Y_i + \left(\frac{Y_j^o - Y_i^o}{\left(d_{ij}^o\right)^2}\right)\Delta X_j + \left(\frac{X_i^o - X_j^o}{\left(d_{ij}^o\right)^2}\right)\Delta Y_j = f_{a_{ij}},$$

where a_{ij} can be substituted for b_{ij} in equations (2.10a) and the remaining terms are defined as in equations (2.06).

2.4 Stochastic Model

The determination of variances, and subsequently the weights of measurements, is known as the stochastic model in a least-squares adjustment.[4] The stochastic model is as important as the functional model since failure to select the stochastic model correctly can affect the ability to isolate blunders in a set of observations and, if grossly incorrect, can bias the adjusted parameters.

The fabric adjustment process assumes that all survey measurement errors are random, normally distributed, and uncorrelated. For normally distributed measurement errors, the precision of the measurement is quantified by the variance σ^2 (or standard deviation σ) of the errors. In surveying, measurement variances are typically formulated based on extensive analysis of instrumentation characteristics, operation, environmental

conditions, and typical field measurement procedures. For a set of uncorrelated observations, a measurement with a high precision, as indicated by a small variance, implies a good observation and in the adjustment should receive a relatively small correction. The converse is also true.

The weight of an observation is a measure of its relative worth compared to other measurements. Weights control the sizes of corrections applied to measurements in an adjustment; specifically, correction size is inversely proportional to the weight. The more precise an observation, the higher its weight, or equivalently, the smaller the variance, the higher the weight. In other words, weights are inversely proportional to variances.[4]

The fabric adjustment provides a default weighting scheme whereby measurement weights are automatically assigned according to the date of survey. Weight factors have been determined after examination of the survey practice at various times and from experience with a wide variety of cadastral data. The measurement technology available at each period in time generally provides a means of assessing the relative accuracy of surveys of differing vintage. The situation in Australia serves as an example to illustrate how this weighting strategy was derived. Though it is possible to alter these weight category values in the fabric, these defaults are recommended for use in most other nations, including United States and Canada.

Prior to 1880, most boundaries were measured with Günter's chain and circumferentor, a compass with sighting vanes mounted on a tripod. In 1880, it became mandatory in Australia to use a theodolite (or transit) with a vernier reading to about 20 seconds of arc. At the same time, the long steel band came into use. This was a flat wire approximately $\frac{1}{8}$" wide and $\frac{1}{64}$" thick with scribed brass ferrules every 10 links and a reader at one end. If used carefully, these steel measuring bands could provide measurement accuracy to better than 1/10,000. Between 1880 and 1900, it seemed that standards gradually improved as surveyors became familiar with this technology. By 1970, electromagnetic distance measurement (EDM) started to make its effect in the cadastral surveying community, along with the replacement of vernier theodolites with optical micrometers and glass circles. Modern survey practice now makes use of total stations with combined EDM and electronic horizontal and vertical circles and the ubiquitous global positioning system (GPS).

A system of automatic weighting based on survey date provides a relatively trouble-free adjustment with no operator intervention necessary. If required, the automatic weighting can be overridden for individual parcels and lines. Here the user can manually set the weights so that individual lines in a parcel have different weights, or a uniform weighting scheme can be defined for an entire plan. The default automatic weighting values are listed in the table below.[2]

Automatic Measurement Weighting Based on Survey Vintage

Category	Bearings, *b* Std. Dev., σ_b (seconds)	Constant, a (meters)	Distances, *d* PPM, b
1—Highest	5	0.001	5
2—After 1980	30	0.01	25
3—1908–1980	60	0.02	50
4—1881–1907	120	0.05	125
5—Before 1881	300	0.20	125
6—1800	3600	1.0	1000
7—Lowest	6000	10.0	5000

Based on the values tabulated, measurement weights are computed as follows:

(2.12a) $\sigma_d^2 = \left[a + \left(b \cdot 10^{-6} d_G\right)\right]^2 \text{ m}^2,$

(2.12b) $w_d = \frac{1}{\sigma_d^2} \text{m}^{-2},$

(2.12c) $w_b = \frac{1}{\sigma_b^2} \text{sec}^{-2},$

where σ_d^2 is the variance of the measured distance; w_d is its corresponding weight; and w_b is the weight of the measured bearing or azimuth. Weights must match the measurement units of their corresponding measurements in the observation equations.

3. Least-Squares Adjustment

3.1 The Least-Squares Principle

Owing to the stochastic nature of measurements, redundant observations are not compatible with the functional model. Any subset of measurements sufficient to determine the unknown parameters will yield different parametric estimates from any other subset.[3] Stated another way, when the linearized functional model has more observation equations than unknown parameters, the solution is overdetermined. Overdetermined systems contain more observations than necessary for the determination of the unknowns. This results in the inability to obtain a unique set of unknown parameters that will satisfy all the observation equations.[14] The problem is solved by replacing the original set of measurements **L** with an estimated set $\hat{\mathbf{L}} = \mathbf{L} + \mathbf{v}$, where **v** is a set of corrections or residuals to the original observations.

The presence of redundant measurements suggests an infinite number of estimates for **v** or $\hat{\mathbf{L}}$ that would satisfy the functional model. Among all the possibilities, there exists one set of estimates that, in addition to being consistent with the functional model, satisfies the fundamental least-squares criterion. The least-squares criterion states that the sum of the squares of the weighted residuals must be a minimum, that is

(3.01) $\phi = \mathbf{v}^t \mathbf{W} \mathbf{v} \rightarrow \text{minimum},$

where **W** is the weight matrix of the observations and **v** is a vector of residuals.[3] This criterion ensures that the new observational estimates $\hat{\mathbf{L}}$ are as close as possible to their measured values, taking their stochastic properties also into account.

3.2 Formulation and Solution of the Least-Squares Problem

3.2.1 Linearization and Iteration

Least-squares treatments are generally performed with linear functional models. Consequently, whenever the equations in the model are nonlinear, some means of linearization must be employed to obtain linear equations. The functional models applied in fabric adjustment are nonlinear (cf. equations [2.05] and [2.07]). These nonlinear models have been approximated with the linear part of a Taylor series expansion resulting in[14]

$$(3.02)\quad \mathbf{F}(\mathbf{X},\mathbf{L}) = \mathbf{F}(\mathbf{X}^o + \Delta\mathbf{X},\mathbf{L}) \cong \mathbf{F}(\mathbf{X}^o,\mathbf{L}) + \left[\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right]_{\mathbf{X}=\mathbf{X}^o} (\Delta\mathbf{X}) \cong \mathbf{0},$$

where $\Delta\mathbf{X} = \mathbf{X} - \mathbf{X}^o$ is the matrix of updates to the estimated parameter vector **X** (cf. [2.06b]). The partial derivatives in (3.02) yield the coefficients in the linearized observation equations (2.06), (2.10), and (2.11). The points of expansion $\mathbf{X}^o$ in (3.02), and therefore in equations (2.06), (2.10), and (2.11), are the parcel corner coordinates obtained from the Helmert transformation described in section 1.3. In matrix form, (3.02) can be written as follows:

$$(3.03)\quad \mathbf{A}\,\Delta\mathbf{X} + \mathbf{f} = \mathbf{0}.$$

This equation is simply the differential form of the original nonlinear functional model and describes the relation of the quantities in the neighborhood of $\mathbf{X}^o$, $\mathbf{L}$, and $\mathbf{f}$ ($\mathbf{f}$ is called the constant or misclosure vector). After least-squares adjustment, the solution vector $\Delta\mathbf{X}$ is obtained, yielding updated unknown parameters $\mathbf{X} = \mathbf{X}^o + \Delta\mathbf{X}$. Often, however, $\mathbf{X}^o$ is not a close enough approximation and $\mathbf{X}^o + \Delta\mathbf{X}$ yields only an improved approximation of the unknowns. The updated vector of unknowns must now be used again in (3.03) and the least-squares solution iterated again to obtain a new vector $\Delta\mathbf{X}$, the elements of which should be smaller than those of the previous one. This process is repeated until the last value of $\Delta\mathbf{X}$ is insignificantly small and the iterative procedure terminates. The final estimate of the unknown parameters $\hat{\mathbf{X}}$ will be the sum of the original approximation $\mathbf{X}^o$ and all the correction vectors $\Delta\mathbf{X}$.

3.2.2 Least-Squares Solution

For a parcel fabric comprising *n* measurements, *m* coordinates, and orientation unknowns, the weighted observation equations in matrix form are given by

$$(3.04)\quad \mathbf{W}\mathbf{L} + \mathbf{W}\mathbf{V} + \mathbf{W}\mathbf{A}\Delta = \mathbf{W}\mathbf{d} \text{ or } \mathbf{W}\mathbf{V} + \mathbf{W}\mathbf{A}\Delta = \mathbf{W}(\mathbf{d} - \mathbf{L}) = \mathbf{W}\mathbf{f},$$

where $\mathbf{d} = \mathbf{F}(\mathbf{X}^o,\mathbf{L})$ are the observation equations evaluated using approximate values of the unknown parameters and $\Delta \equiv \Delta\mathbf{X}$. The matrices in (3.04) have the following form:[9]

$$(3.04a)\quad \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ & & & \\ \ldots & \ldots & \ldots & \ldots \\ & & & \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_m \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ \cdot \\ \cdot \\ \cdot \\ l_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix},$$

with weight matrix

(3.04b) $$\begin{bmatrix} w_{11} & & & & \\ & w_{22} & & \mathbf{0} & \\ & & w_{33} & & \\ & & & \ddots & \\ & \mathbf{0} & & & \\ & & & & w_{nn} \end{bmatrix},$$

and where

$v_1, v_2, ..., v_n$ are the residuals for the *n* measurements;

$a_{11}, a_{12}, ..., a_{nm}$ are the numerical coefficients of the unknown parameters (from [2.06], [2.10], and [2.11]);

$\Delta_1, \Delta_2, ..., \Delta_m$ are the *m* unknown parameters or updates to them (cf. [2.06b]);

$d_1, d_2, ..., d_n$ are numerical constants obtained by evaluating the observation equations at the expansion points, that is, the approximate coordinates of the parcel corners (cf. [2.14] and [2.06a], [2.10a], and [2.11]);

$l_1, l_2, ..., l_n$ are the numerical values of the *n* measurements;

$f_1, f_2, ..., f_n$ are numerical constants on the right-hand side of the observation equations, that is, $d_i - l_i$ (cf. [2.14] and [2.06a], [2.10a], and [2.11]); and

$w_{11}, w_{22}, ..., w_{nn}$ are the uncorrelated weights of the *n* measurements. Imposing the least-squares condition of (3.01) on (3.04), it can be shown that the least-squares solution is given by[3,8]

(3.05) $\mathbf{\Delta} = (\mathbf{A}^t\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^t\mathbf{W}\mathbf{f}$,
letting

$\mathbf{N} = \mathbf{A}^t\mathbf{W}\mathbf{A}$ and $\mathbf{t} = \mathbf{A}^t\mathbf{W}\mathbf{f}$,

(3.05) becomes

(3.06) $\mathbf{\Delta} = \mathbf{N}^{-1}\mathbf{t}$.

In (3.06), **N** is the $m \times m$ matrix of normal equations, **t** is the $m \times 1$ vector of constants, and **Δ** is the $m \times 1$ solution vector of corrections to the approximate coordinates and unknown orientations. Having obtained **Δ**, the adjusted observations and residuals are obtained directly from

(3.07) $\hat{\mathbf{L}} = \mathbf{L} + \mathbf{f} - \mathbf{A}\mathbf{\Delta}$

and

(3.08) $\hat{\mathbf{v}} = \hat{\mathbf{L}} - \mathbf{L} = \mathbf{f} - \mathbf{A}\boldsymbol{\Delta}$.

If observation weights have been chosen correctly, then the precision of the estimated parameters is obtained from the post-adjustment covariance matrix, that is,

(3.09) $\mathbf{C}_{\boldsymbol{\Delta}} = \mathbf{C}_{\hat{\mathbf{X}}} = \mathbf{N}^{-1}$.

An unbiased estimate of the *a posteriori* reference variance after adjustment is obtained from

(3.10) $\hat{\sigma}_o^2 = \dfrac{\mathbf{v}^t \mathbf{W} \mathbf{v}}{r}$,

where $r = n - m$ is the number of degrees of freedom (or redundancy) in the network, defined as the number of observations *n* minus the number of unknown parameters *m*.

It is often the case in adjustments that actual observation weights are replaced by relative weights, or the weights chosen are biased by some unknown factor. In these cases, the weight coefficient is taken as the reference variance given by (3.10) and the actual weight of an observation is obtained from $w_i = \hat{\sigma}_o^2 / \sigma_i^2$. When weights are chosen correctly, $\hat{\sigma}_o^2$ should equal unity. If this is not the case, then the covariance matrix in (3.09) should be scaled by the reference variance $\hat{\sigma}_o^2$ to obtain the correct precision of the estimated parameters.

3.2.3 Iteration Termination for Linearized Least-Squares

The following discussion is important for users of the fabric adjustment, as the decision to repeat an iteration of the adjustment is left entirely to the user's discretion. The released software does not allow the process to iterate automatically.

As mentioned in section 3.2.1 and implied in section 2.3, the functional models of this adjustment are nonlinear and must be linearized using the Taylor series expansion given by (3.02). This means that the least-squares equation in (3.06) must undergo repeated recomputation of the nonlinear observation equations until some satisfactory solution is reached. To terminate this iterative process, a criterion, or criteria, must be defined by which to ascertain that a solution has converged sufficiently close to the theoretical nonlinear solution. The chosen criterion should be a representative indicator of convergence and possess high discriminative power to signalize the attainment of convergence. Several methods present themselves: [3,4]

- **Maximum increment method:** This method involves monitoring the absolute magnitude of the increments to the coordinate parameters (Δ) or functions of such increments. Once the stated condition is satisfied, convergence has occurred and iteration is stopped. There are several ways to accomplish this.

  (a) After sufficient iterations of (3.06), the value of each increment (Δ) should approach zero. A negligibly small threshold ε may be used as a criterion for iteration termination:

(3.11a) $\left|\mathbf{\Delta}\right| < \varepsilon$.

(b) Instead of requiring that each increment approach zero, it may be sufficient to check the maximum increment:

(3.11b) $\left|\mathbf{\Delta}_{\max}\right| < \varepsilon$.

(c) Another criterion may involve testing a function of the coordinate increments instead of individual increments such as

(3.11c) $\Delta\hat{X}^2 + \Delta\hat{Y}^2 < \varepsilon^2$.

This test is particularly interesting as it resembles a type of *spatial convergence*. Convergence is attained when the radial increment to each point is such that it lies within a negligibly small circle of radius ε. A maximum radial increment implementation similar to (3.11b) is also a possibility.

- **Maximum residual method:** When all residuals achieve their maximum value, or equivalently, when the absolute change in each residual between iterations becomes negligibly small, then convergence has occurred, that is,

(3.11d) $\left|\mathbf{v}_{k-1} - \mathbf{v}_k\right|_j < \varepsilon_j$.

Here *k* is the iteration number and *j* the observation type, suggesting different thresholds for different kinds of observations (e.g., distances vs. bearings). This criterion suits the fabric adjustment particularly well since residuals form the singular indicator of the quality and success of the adjustment. The test itself, however, may not be sensitive to gross errors since the residuals of even the bad observations may have already reached their maximum value and condition (3.11d) will be satisfied. Of course, examination of the actual values of the residuals should indicate the presence of the blunder and, for well-conditioned networks, the culprit observation. Conversely, where multiple blunders exist because of a flawed functional model or an unrealistic stochastic model, the adjustment may fail to converge based on criteria (3.11d) even after many iterations.

- **Monitoring the reference variance:** Since the least-squares method converges quadratically, the iterative process should definitely be stopped if the reference variance of (3.10) increases. An increasing reference variance suggests a diverging solution. By monitoring the change in reference variance from one iteration to the next, convergence or divergence can be detected. It is assumed convergence has occurred when the change in reference variance falls below some threshold value and may be tested by[3]

(3.11e) $\left|\frac{\hat{\sigma}_o^{k-1} - \hat{\sigma}_o^{k}}{\hat{\sigma}_o^{k-1}}\right| = \left|1 - \frac{\hat{\sigma}_o^{k}}{\hat{\sigma}_o^{k-1}}\right| < \delta$,

where *k* is the iteration number. According to Mikhail (1976),[3] a suitable value for δ is 1 percent between iterations. If the reference variance increases, the solution is diverging and the iteration process should be terminated. Divergence can happen in one of two ways: (1) a gross error exists in the data, or (2) the maximum residual size is less than the precision of the measurements. In the second case, the solution has already converged, and when another iteration is attempted, the solution will

converge only to diverge on the next iteration. This apparent oscillation between convergence and divergence in successive solutions is caused either by round-off errors or by convergence limits that are too stringent for the quality of the data. Nevertheless, monitoring changes in the reference variance will always indicate convergence or divergence in the least-squares solution[4]. On the other hand, Mikhail (1976)[3] asserts that testing convergence based on reference variance criteria can be unreliable in cases of adjustments with low numerical stability such as severely ill-conditioned survey networks. Since parcel fabric networks tend to be fairly well conditioned, (3.11e) may serve as a useful convergence criterion, ideally in conjunction with the maximum residual method of (3.11d).

- **Limited iteration method:** One very simple method of iteration termination involves limiting the number of iterations to a predefined maximum. The drawback of this method is that if this maximum is too low, convergence may not be attained at the termination of the process, and if it is too high, time is wasted on unnecessary iterations. Moreover, gross data errors can significantly affect both convergence and the iterations such that, in the presence of gross errors, it is possible to never reach convergence regardless of the number of iterations.

  After every manually executed iteration, the fabric adjustment process reports the following convergence indicators:

  - The increment $\mathbf{\Delta}_i$ estimated for each coordinate
  - The maximum increment $\mathbf{\Delta}_{\max}$ applied
  - The average increment $\frac{\mathbf{\Delta}}{m}$

3.2.4 Post-adjustment Quality Indicators

The parcel fabric adjustment does not calculate the post-adjustment covariance matrix in (3.09) or the *a posteriori* reference variance in (3.10). Instead, the software reports metrics to indicate the geometric distortion of each parcel with respect to its measured values. The results of the least-squares adjustment of a fabric comprise an adjustment summary, the results of pre-processing tolerance tests, adjusted cadastral corner coordinates, residual statistics, and statistics indicating the agreement between the adjusted values and the parcel measurements. For each parcel, linear and angular misclosure is given along with estimates of scale and rotation between the adjusted and measured parcel lines.

3.3 Computational Methods and Optimization

3.3.1 Generating the Normal Equations

Adjustment of a large parcel fabric can lead to a very large system of normal equations, so any means to reduce in-memory computer space aids computational efficiency. Taking advantage of the structured form of a parcel fabric system of equations, the fabric adjustment uses a well-known systematic procedure to formulate the normal equations directly, one observation at a time.
If the observations are independent and uncorrelated, then the weight matrix **W** will be diagonal and the intermediate stage involving formation of the **A, W, A^{t}W,** and **f** matrices in (3.05) is unnecessary. It can be shown that the structure of the normal equations has the form[4]

$$\left(\sum_{i=1...n} w_i a_i^2\right)\Delta_1 + \left(\sum_{i=1...n} w_i a_i b_i\right)\Delta_2 + \left(\sum_{i=1...n} w_i a_i c_i\right)\Delta_3 + ... + \left(\sum_{i=1...n} w_i a_i n_i\right)\Delta_m = \left(\sum_{i=1...n} w_i a_i l_i\right)$$

$$\left(\sum_{i=1...n} w_i b_i a_i\right)\Delta_1 + \left(\sum_{i=1...n} w_i b_i^2\right)\Delta_2 + \left(\sum_{i=1...n} w_i b_i c_i\right)\Delta_3 + ... + \left(\sum_{i=1...n} w_i b_i n_i\right)\Delta_m = \left(\sum_{i=1...n} w_i b_i l_i\right)$$

(3.12)
$$\left(\sum_{i=1...n} w_i c_i a_i\right)\Delta_1 + \left(\sum_{i=1...n} w_i c_i b_i\right)\Delta_2 + \left(\sum_{i=1...n} w_i c_i^2\right)\Delta_3 + ... + \left(\sum_{i=1...n} w_i c_i n_i\right)\Delta_m = \left(\sum_{i=1...n} w_i c_i l_i\right)$$

$$\vdots$$

$$\left(\sum_{i=1...n} w_i n_i a_i\right)\Delta_1 + \left(\sum_{i=1...n} w_i n_i b_i\right)\Delta_2 + \left(\sum_{i=1...n} w_i n_i c_i\right)\Delta_3 + ... + \left(\sum_{i=1...n} w_i n_i^2\right)\Delta_m = \left(\sum_{i=1...n} w_i n_i l_i\right),$$

where

$a_i, b_i, ..., n_i$ are numerical coefficients of the unknown parameters;

$\Delta_1, \Delta_2, ..., \Delta_m$ are the *m* unknown parameters, or updates to them;

$l_1, l_2, ..., l_n$ are the numerical values of the *n* measurements; and

$w_1, w_2, ..., w_n$ are the uncorrelated weights of the *n* measurements.

In (3.12), the coefficients of the Δs are the elements of the normal equation matrix **N** of (3.06), and the terms on the right-hand side are the elements of the constant vector **t.** As each observation is added to the fabric network, first its coefficient is computed from (2.06), (2.10), or (2.11), then its contribution to the corresponding coefficient of the normal equations is computed individually from (3.12) and subsequently added. The constant vector is built up in similar fashion and the process is repeated until all observations have been considered. The schematic algorithm is developed as follows:

Normal Equations Generation - Schematic Algorithm

Step 1: Zero the normal equation and constant vector arrays.

Step 2: Zero a single row of the observation equation coefficient array and constant vector array.

Step 3: Compute the values of a single row of the observation equation and constant matrices using (2.06), (2.10), and (2.11), then add the values to the appropriate normal and constant array elements in (3.12).

Step 4: Repeat steps 2 and 3 for all observations.

3.3.2 Cholesky Decomposition

From (3.12), the normal equation matrix is symmetric and has the form

$$(3.13)\quad \mathbf{N} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\ & \bullet & \bullet & \bullet & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} \end{bmatrix}, \quad a_{ij} = a_{ji} \text{ for } i \neq j.$$

Symmetric matrices afford significant computational advantages when solving a system of linear equations. An advantage is that one only needs to store the upper (or lower) triangular portion of the normal equation matrix; another is that the solution of the system does not require inverting the normal equation matrix **N,** a computationally demanding process. The fabric adjustment solves the system (3.12) for the unknown parameters using the method of Cholesky decomposition, a special case of a broader class of triangular decomposition methods for the direct solution of linear systems of equations. Cholesky decomposition takes advantage of the fact that **N** is always positive definite*. For any positive-definite matrix **A**, there is a unique lower triangular matrix **L** with positive diagonal elements such that $\mathbf{A} = \mathbf{L}\mathbf{L}^t$.[15] Hence, the normal equation matrix (3.13) can be written as[4]

$$(3.14)\quad \mathbf{N} = \mathbf{L}\mathbf{U} = \mathbf{L}\mathbf{L}^t = \mathbf{U}^t\mathbf{U},$$

where **L** is a lower triangular matrix of the form

$$(3.14a)\quad \mathbf{L} = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ & \bullet & \bullet & \bullet & \\ l_{m1} & l_{m2} & l_{m3} & \dots & l_{mm} \end{bmatrix}.$$

The Cholesky decomposition formulas for determining the l_{ij} in the $\mathbf{L}^t$ matrix are[16]

$$(3.15a)\quad a_{11} \leftarrow l_{11} = \sqrt{a_{11}}; \qquad a_{1j} \leftarrow l_{1j} = \frac{a_{1j}}{l_{11}},$$

$$(3.15b)\quad a_{ii} \leftarrow l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ki}}, \; i = 2,3,\dots,m; \qquad a_{ij} \leftarrow l_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ki}l_{kj}}{l_{ii}}, \quad j = 2,3,\dots,m,$$

$$(3.15c)\quad a_{ij} \leftarrow l_{ij} = 0, \quad i > j.$$

The schematic algorithm for Cholesky decomposition using (3.15a–c) is as follows:

Cholesky Decomposition - Schematic Algorithm

Step 1: Compute row 1 of **L** using (3.15a) and replace a_{11} with l_{11} in the normal

* A matrix **A** is said to be positive definite if $\mathbf{x}^t\mathbf{A}\mathbf{x} > 0$ for every $\mathbf{x} > 0$.

equation array (i.e., $a_{11} \leftarrow l_{11}$).

Step 2: Compute the diagonal term l_{ii} of the next row of **L** (i.e., row 2) using the first equation of (3.15b) and replace $a_{ii} \leftarrow l_{ii}$.

Step 3: Compute the remaining terms of the row using the second equation of (3.15b) and replace $a_{ij} \leftarrow l_{ij}$.

Step 4: Repeat steps 2 and 3 until all *m* rows are computed.

Factoring the normal equation matrix into triangular matrices **L** and $\mathbf{L}^t$ has the advantage that the least-squares solution of (3.06) can be obtained without inverting the **N** matrix. Rewriting (3.06) in the form

(3.16) $\mathbf{N\Delta} = \mathbf{LL}^t\mathbf{\Delta} = \mathbf{t}$,

which can be written as

(3.17) $\mathbf{Ly} = \mathbf{t}$, where $\mathbf{L}^t\mathbf{\Delta} = \mathbf{y}$; then

(3.18) $\mathbf{y} = \mathbf{L}^{-1}\mathbf{t}$ and $\mathbf{\Delta} = \left(\mathbf{L}^t\right)^{-1}\mathbf{y}$.

Taking advantage of the triangular form of **L,** the solution for the elements of the vector **y** is obtained directly by forward substitution as follows:[15,16]

$$(3.19)\quad t_1 \leftarrow y_1 = \frac{t_1}{l_{11}}, \quad t_i \leftarrow y_i = \frac{t_i - \sum_{k=1}^{i-1} l_{ik} y_k}{l_{ii}}, \quad i = 2, 3, ..., m.$$

Having determined **y,** the solution for **Δ** is obtained from the second equation of (3.18) in similar manner to that above. However, this time the solution begins in the lower right corner and proceeds up the matrix $\mathbf{L}^t$**.** This is called back substitution and proceeds as follows:[15,9]

$$(3.20)\quad y_m \leftarrow \Delta_m = \frac{y_m}{l_{mm}}, \quad y_k \leftarrow \Delta_k = \frac{y_k - \sum_{j=k+1}^{m} l_{kj}\Delta_j}{l_{kk}}, \quad k = m-1, m-2, ..., 1.$$

Solution of Normal Equations - Schematic Algorithm

Step 1: Compute y_1 using the first equation of (3.19) and replace in the **t** vector, $t_1 \leftarrow y_1$.

Step 2: Compute each successive y_i using the second equation of (3.19) and replace, $t_i \leftarrow y_i$.

Step 3: Compute Δ_m using the first equation of (3.20) and replace in the **y** vector, $y_m \leftarrow \Delta_m$.

Step 4: Compute each successive Δ_k using the second equation of (3.20) and $y_k \leftarrow \Delta_k$.

3.3.3 Optimization of Sparse Matrices

A sparse matrix is a matrix populated primarily with zeros. When storing and manipulating sparse matrices on the computer, it is often necessary to modify the standard algorithms and take advantage of the sparse structure of the matrix. Sparse data is, by its nature, easily compressed and can yield enormous savings in memory usage.

Once the normal equations are generated using (3.12), the elements of the normal equation and constant matrices, **N** and **t,** are available. Generally, the normal equation matrix will contain more zero elements than nonzero ones. This sparse **N** matrix needs its elements reordered to reduce its bandwidth* for computational efficiency. The fabric adjustment uses the following process.

The first step involves sorting the parcels according to their centroid coordinates. This brings the parcels into spatial juxtaposition so that, in most cases, input measurements from any given parcel are selected for processing soon after those from a nearby parcel. As each parcel is selected and the measurements are processed into normal equations, index numbers are allocated to the unknowns yielding a first attempt at bandwidth minimization. After forming all the equations, the algorithm finds and selects the equation with the largest separation between coefficients. Renumbering one or both parcel points in that equation, it is then examined to see if the renumbering has reduced the separation. If it has, the process is repeated for the equation with the next largest coefficient separation until renumbering is no longer possible.[6]

During generation of the normal equations, only the upper triangular part of the normal equation matrix of (3.13) is stored, and each row is limited in length to the bandwidth. This two-dimensional matrix is stored as a single one-dimensional vector. This storage scheme requires a mathematical mapping between the original matrix element indices and their corresponding indices in the vector. Any element U_{ij} of the upper triangular matrix corresponds to the vector element V_i at index[4]

$$(3.21)\quad U_{ij} \equiv V\left[\frac{j(j-1)}{2} - i\right] = V[i].$$

3.4 Least-Squares Adjustment for parcel fabrics

To create a meaningful cadastral dataset, it is important to understand parcel fabric topology and build the adjustment solution according to processes unique to this type of survey data. The parcel fabric allows a parcel-based analysis and adjustment model and simulates a cadastral surveying paradigm in its sequence and implementation of computational processes. Some of these unique implementations will now be described.

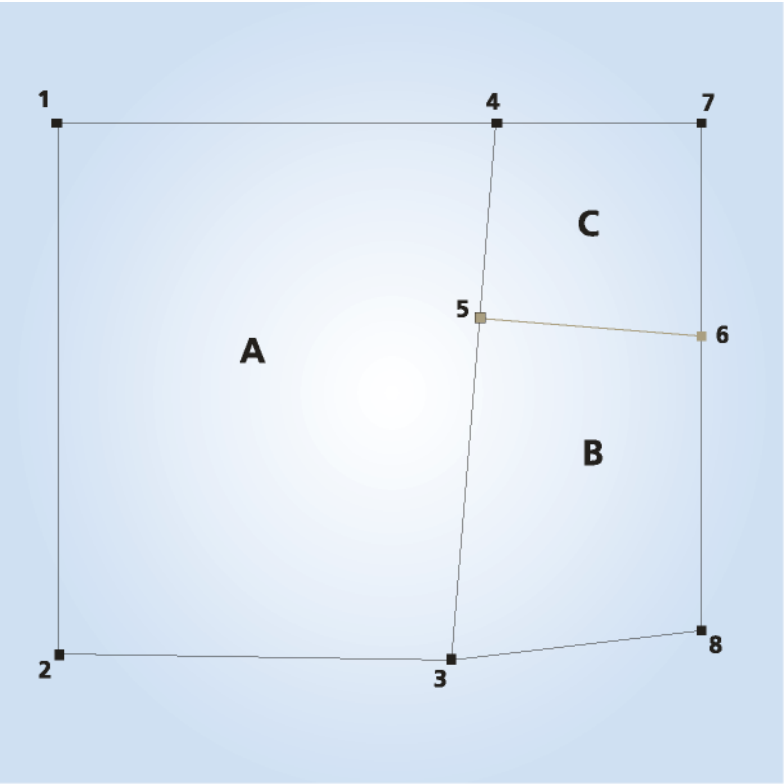
- **Observation equations:** For each parcel, a bearing (or azimuth) and a distance observation equation are written for each line of the parcel. This implies that common lines between parcels receive multiple observation equations in the adjustment. For the bearing equations, each parcel is treated as a single unit, and an orientation term for the parcel is included in the bearing observation equations.

- **Line points:**[6] Line points are defined as points that lie on a parcel boundary line but do not form a boundary corner for the parcel. Linear dimensions may or may not

* The bandwidth of a matrix is computed as the maximum of the set of bandwidths of each row of the matrix. The bandwidth of a row of the matrix is essentially the number of matrix elements between the first and last nonzero elements in the row, with the proviso that the diagonal entry is always treated as though it were nonzero. Given a sparse $n \times m$ matrix **A**, the bandwidth for the matrix is defined as $B(\mathbf{A}) = \left\langle \max |i-j| \,\middle|\, a_{ij} \neq 0 \right\rangle$.[17]

exist between the line point and the parcel corners on either side of the line point. The following example illustrates the concept of a line point. Consider figure 5 below.

Figure 5
Line Points



Suppose that parcel A (defined by points 1, 2, 3, and 4) and B (defined by points 3, 4, 7, and 8) were created prior to parcel C. Later, parcel C, comprising lot lines 4–5, 5–6, 6–7, and 7–4, is surveyed to subdivide parcel B. Further suppose the measured length of line 7–4 of parcel C is less than the length of the same line for parcel B. The effect is for point 5 to be pulled off line 3–4. Now, it is known that point 5 lies on line 3–4, but is that line straight? If point 3 was not visible from 4, the surveyor may have simply turned off the original angle at 4 to place 5. The surrounding parcels will also have an effect on the final position of all corners. To establish the condition that point 5 lies on line 3–4, two bearing equations will be included, in addition to the bearing equation for line 3–4, to force the line point 5 onto the existing parcel line. The table below shows the bearing observation equations used for parcels A, B, and C and for line point 5.

Parcel Bearing Equations			Line Point 5	
Parcel A	**Parcel B**	**Parcel C**	**Bearing Equations**	**Source**
a_{12}	B_{43}	c_{45}	l_{54}	Bearing of a_{34}
a_{23}	B_{38}	c_{56}		
a_{34}	B_{87}	c_{67}	l_{53}	Bearing of a_{43}
a_{41}	B_{74}	c_{74}		

After the adjustment, point 5 may or may not be exactly on line 3–4. If it has been decided that the line should be straight an option in fabric adjustment, when set, will move the line point back onto the line after the adjustment. Since the least-squares adjustment yields the most likely position of the parcel corners, this option is often used for those line points that are only a small distance off the line. The particular mathematical approach described here is adapted to get a practical and reasonable solution when the user may not have full control or full understanding of the data.

- **Iteration termination:** The fabric adjustment process does not implement an automatic convergence criterion. The user is expected to assess the quality of an adjustment using the reported indicators (see section 3.2.3) after each iteration, and then decide when satisfactory results have been achieved. If necessary, the user can perform additional adjustment iterations, assessing the results after each one. Adjustment iterations are then manually implemented and terminated.

Conclusion

The mathematical models used in the ArcGIS Parcel Fabric least-squares adjustment module have been described. The algorithms used are rigorous and yield adjusted coordinates similar to those of other survey data adjustment software.

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